PHYSICAL REVIEW B **80**, 224406 (2009)

Translation-symmetry-protected topological orders in quantum spin systems

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(Received 10 August 2009; published 4 December 2009)

In this paper we systematically study a simple class of translation-symmetry protected topological orders in quantum spin systems using slave-particle approach. The lattice spin systems are translation invariant, but may break any other symmetries. We consider topologically ordered ground states that do not spontaneously break any symmetry. Those states can be described by *Z2A* or *Z2B* projective symmetry group. We find that the *Z2A* translation symmetric topological orders can still be divided into 16 subclasses corresponding to 16 translation-symmetry protected topological orders. We introduced four Z_2 topological indices $\zeta_{\mathbf{k}}=0,1$ at $\mathbf{k}=(0,0)$, $(0,\pi)$, $(\pi,0)$, and (π,π) to characterize those 16 topological orders on square lattice. We calculated the topological degeneracies and crystal momenta for those 16 topological phases on even-by-even, even-by-odd, odd-by-even, and odd-by-odd lattices, which allows us to physically measure such topological orders. We predict the appearance of gapless fermionic excitations at the quantum phase transitions between those symmetry protected topological orders. Our result can be generalized to any dimensions. We find 256 translation-symmetry protected *Z2A* topological orders for a system on three-dimensional cubic lattice.

DOI: 10.1103/PhysRevB.80.224406 PACS number(s): 75.45.+j, 71.27.+a

I. INTRODUCTION AND MOTIVATION

For long time, people believe that Landau symmetry breaking theory[1] and the associated local order parameters[2,3] describe all kinds of phases and phase transitions. However, in last 20 years, it became more and more clear that Landau theory cannot describe all quantum states of matter (the states of matter at zero temperature).[4,5] A nontrivial example of states of matter beyond Laudau theory is the fractional quantum Hall (FQH) states.[6] FQH states do not break any symmetry and hence cannot be described by broken symmetries. The subtle structures that distinguish different FQH states are called topological order.[4,5,7] Physically, topological order describes the internal order (or more precisely, the long range entanglement) in a gapped quantum ground state. It can be (partially) characterized by robust ground-state degeneracy.[4,8] Recently, many different systems with topologically ordered ground states were found.[8–16]

Quantum spin liquid states in general contain nontrivial topological orders. In the projective construction of spin liquid (also known as slave-particle approach),[17–23] there exist many-spin liquids with low energy SU(2), U(1), or Z_2 gauge structures. Those spin liquids all have exactly the same symmetry. To distinguish those spin liquids, we note that although the spin liquids have the same symmetry, within the projective construction, their ansatz are not directly invariant under the translations. The ansatz are invariant under the translations followed by different gauge transformations. So the invariant groups of the ansatz are different. We can use the invariant group of the ansatz to characterize the order in the spin liquids. The invariant group of an ansatz is formed by all the combined symmetry transformations and the gauge transformations that leave the ansatz invariant. Such a group is called the projective symmetry group (PSG).[24] Thus although one cannot use symmetry and order parameter to describe different orders in the spin liquids, one can use the PSG to characterize/distinguish the different quantum/topological orders of spin liquid states.

The simplest kind of topological orders is the Z_2 topological order where the slave-particle ansatz is invariant under a Z_2 gauge transformation. According to the PSG characterization within the projective construction, for system with only lattice translation-symmetry, there can be two different classes of Z_2 topological orders labeled by *Z2A* and *Z2B* (a *Z2B* ansatz has π flux going through each plaquette).[24] In this paper, we will study the *Z2A* topological orders and ask "are there distinct *Z2A* topological orders?" We find that there are indeed distinct *Z2A* topological orders. They can be labeled by four Z_2 topological indices $\zeta_{\mathbf{k}}=0,1$ at $\mathbf{k}=(0,0)$, $(0,\pi)$, $(\pi,0)$, and (π,π). So the $\zeta_{\mathbf{k}}$ characterization is beyond the PSG characterization of quantum/topological order and provides additional information for translation-symmetry protected Z_2 topological order.

II. GENERAL "MEAN-FIELD" FERMION HAMILTONIAN OF Z_2 TOPOLOGICAL ORDERS

We will use the projective construction (the slave-particle theory)[17,23] to systematically construct different translation symmetric Z_2 topological orders in spin-1/2 systems on square lattice. In such a construction, we start with "mean-field" fermion Hamiltonian[24]

$$H_{\text{mean}} = \sum_{\mathbf{ij}} \psi_{\mathbf{i}}^{\dagger} u_{\mathbf{ij}} \psi_{\mathbf{j}} + \sum_{\mathbf{ij}} (\psi_{\mathbf{i}}^{\dagger} \eta_{\mathbf{ij}} \psi_{\mathbf{j}}^{\dagger} + \text{H.c.}) + \sum_{\mathbf{i}} \psi_{\mathbf{i}}^{\dagger} a_{\mathbf{i}} \psi_{\mathbf{i}}, \quad (1)$$

where $u_{\mathbf{ij}}$, $\eta_{\mathbf{ij}}$, and $a_{\mathbf{i}}$ are 2 by 2 complex matrices. The η term is included since our spin-1/2 systems in general do not have any spin rotation symmetry. We like to mention that $a_{\mathbf{i}}$ are not free parameters. $a_{\mathbf{i}}$ should be chosen such that

$$\langle \Psi_{\text{mean}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})} | \psi_{\mathbf{i}}^{\dagger} \sigma^{l} \psi_{\mathbf{i}} | \Psi_{\text{mean}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})} \rangle = 0, \quad l = 1,2,3, \tag{2}$$

where σ^l are the Pauli matrices. In this paper, we will only consider translation invariant ansatz $u_{\mathbf{ij}} = u_{\mathbf{i+a,j+a}}$ and $\eta_{\mathbf{ij}} = \eta_{\mathbf{i+a,j+a}}$. Those states are characterized by *Z2A* PSG and are *Z2A* topological states.[24]

Let $|\Psi_{\text{mean}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})}\rangle$ be the ground state of H_{mean}. Then a many-spin state can be obtained from the mean-field state $|\Psi_{\text{mean}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})}\rangle$ by projection

$$|\Psi_{\text{spin}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})}\rangle = \mathcal{P} |\Psi_{\text{mean}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})}\rangle, \tag{3}$$

into the subspace with even numbers of fermion per site. Here the projection operator is

$$\mathcal{P} = \prod_{\mathbf{i}} \frac{1 + (-1)^{n_{\mathbf{i}}}}{2},$$

and $n_{\mathbf{i}} = \psi_{\mathbf{i}}^{\dagger} \psi_{\mathbf{i}}$ is fermion operator at site $\mathbf{i}$.

We note that after the projection, each site can have either no fermion or two fermions. If we associate the no fermion state as the spin-down state and the two-fermion state as the spin-up state, then the projected state $|\Psi_{\text{spin}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})}\rangle$ can be viewed as a quantum state for the spin system. This is how we construct many-spin state from the mean-field Hamiltonian. For each choice of the ansatz $(u_{\mathbf{ij}}, \eta_{\mathbf{ij}}, a_{\mathbf{i}})$, this procedure produces a physical many-spin wave function $|\Psi_{\text{spin}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})}\rangle$. So the ansatz $(u_{\mathbf{ij}}, \eta_{\mathbf{ij}})$ can also be viewed as a set of labels that label a many-spin state and $|\Psi_{\text{spin}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})}\rangle$ can be viewed as a trial wave function for a spin-1/2 system, with $u_{\mathbf{ij}}$ and $\eta_{\mathbf{ij}}$ being variational parameters.

For the translation invariant ansatz, one can rewrite the mean-field fermion Hamiltonian in momentum space by presenting

$$\Psi_{\mathbf{k}} = \begin{pmatrix} \psi_{1,\mathbf{k}} \\ \psi_{1,-\mathbf{k}}^{\dagger} \\ \psi_{2,\mathbf{k}} \\ \psi_{2,-\mathbf{k}}^{\dagger} \end{pmatrix},$$

and

$$\Psi_{\mathbf{k}}^{\dagger} = (\psi_{1,\mathbf{k}}^{\dagger} \quad \psi_{1,-\mathbf{k}} \quad \psi_{2,\mathbf{k}}^{\dagger} \quad \psi_{2,-\mathbf{k}}).$$

Note that $\Psi_{\mathbf{k}}$ satisfy the following algebra

$$\{\Psi_{I\mathbf{k}}^{\dagger}, \Psi_{J\mathbf{k}'}\} = \delta_{IJ}\delta_{\mathbf{k}-\mathbf{k}'}, \quad \{\Psi_{I\mathbf{k}}, \Psi_{J\mathbf{k}'}\} = \Gamma_{IJ}\delta_{\mathbf{k}+\mathbf{k}'},$$

where

$$\Gamma = \sigma_1 \otimes \sigma_0 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \tag{4}$$

We also note that $(\Psi_{-\mathbf{k}}^{\dagger}, \Psi_{-\mathbf{k}})$ can be expressed in term of $(\Psi_{\mathbf{k}}^{\dagger}, \Psi_{\mathbf{k}})$,

$$\Psi_{-\mathbf{k}} = \Gamma \Psi_{\mathbf{k}}^{*}, \quad \Psi_{-\mathbf{k}}^{\dagger} = \Psi_{\mathbf{k}}^{T} \Gamma. \tag{5}$$

In terms of $\Psi_{\mathbf{k}}$, H_{mean} can be written as

$$H_{\text{mean}} = \sum_{\mathbf{k}\neq 0} \Psi_{\mathbf{k}}^{\dagger} M(\mathbf{k}) \Psi_{\mathbf{k}} + \sum_{\mathbf{k}=0} \Psi_{\mathbf{k}}^{\dagger} M(\mathbf{k}) \Psi_{\mathbf{k}}, \tag{6}$$

where $-\pi < k_x, k_y < +\pi$, and $M(\mathbf{k})$ are 4×4 Hermitian matrices $M(\mathbf{k}) = M^{\dagger}(\mathbf{k})$. Here $\mathbf{k}=0$ means that $(k_x,k_y)=(0,0)$, $(0,\pi)$, $(\pi,0)$, or (π,π). Also k_x and k_y are quantized: $k_x = \frac{2\pi}{L_x} \times$ integer and $k_y = \frac{2\pi}{L_y} \times$ integer, where L_x and L_y are size of the square lattice in the x and y directions. Note that on an even-by-even lattice (i.e., L_x=even and L_y=even), $(k_x,k_y) = (0,0)$, $(0,\pi)$, $(\pi,0)$, or (π,π) all satisfy the quantization conditions $k_x = \frac{2\pi}{L_x} \times$ integer and $k_y = \frac{2\pi}{L_y} \times$ integer. In this case, $\Sigma_{\mathbf{k=0}}$ sums over all the four points $(k_x,k_y)=(0,0)$, $(0,\pi)$, $(\pi,0)$, and (π,π). On other lattices, $\Sigma_{\mathbf{k=0}}$ sums over less points. Say on an odd-by-odd lattice, only $(k_x,k_y)=(0,0)$ satisfies the quantization conditions $k_x = \frac{2\pi}{L_x} \times$ integer and $k_y = \frac{2\pi}{L_y} \times$ integer. In this case, $\Sigma_{\mathbf{k=0}}$ sums over only $(k_x,k_y) = (0,0)$ point.

We note that

$$\Psi_{\mathbf{k}}^{\dagger}\Psi_{\mathbf{k}} = 2, \quad \Psi_{\mathbf{k}}^{\dagger}\sigma_0 \otimes \sigma_3 \Psi_{\mathbf{k}} = 0.$$

Thus up to a constant in H_{mean}, we may assume $M(\mathbf{k})$ to satisfy

$$\text{Tr}\, M(\mathbf{k}) = 0, \quad \text{Tr}[M(\mathbf{k})\sigma_0 \otimes \sigma_3] = 0. \tag{7}$$

Due to Eq. (5),

$$\Psi_{-\mathbf{k}}^{\dagger} M(-\mathbf{k}) \Psi_{-\mathbf{k}} = \text{Tr}\, M(\mathbf{k}) - \Psi_{\mathbf{k}}^{\dagger} \Gamma M^{T}(-\mathbf{k}) \Gamma \Psi_{\mathbf{k}}.$$

Thus, we may rewrite Eq. (6) as

$$H_{\text{mean}} = \sum_{\mathbf{k}>0} \Psi_{\mathbf{k}}^{\dagger} U(\mathbf{k}) \Psi_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{k}=0} \Psi_{\mathbf{k}}^{\dagger} U(\mathbf{k}) \Psi_{\mathbf{k}},$$

$$U(\mathbf{k}) = M(\mathbf{k}) - \Gamma M^{T}(-\mathbf{k}) \Gamma. \tag{8}$$

Here $\mathbf{k} > 0$ means that $\mathbf{k} \neq 0$ and $k_y > 0$ or $k_y = 0$ and $k_x > 0$. Clearly $U(\mathbf{k})$ satisfy

$$U(\mathbf{k}) = -\Gamma U^{T}(-\mathbf{k}) \Gamma, \quad U(\mathbf{k}) = U^{\dagger}(\mathbf{k}).$$

Now we expand $U(\mathbf{k})$ by 16 Hermitian matrices

$$M_{\{\alpha\beta\}} \equiv \sigma_{\alpha} \otimes \sigma_{\beta}, \quad \alpha, \beta = 0,1,2,3, \tag{9}$$

where $\sigma_0 = \mathbf{1}$. We have

$$U(\mathbf{k}) = \sum_{\{\alpha,\beta\}} c_{\{\alpha\beta\}}(\mathbf{k}) M_{\{\alpha\beta\}},$$

where $c_{\{\alpha\beta\}}(\mathbf{k})$ are real. The 16 4×4 matrices $M_{\{\alpha\beta\}}$ can be divided into two classes: in one class, the matrices satisfy

$$M = -\Gamma M^{T} \Gamma.$$

We call them "*even matrices*;" in the other class, the matrices satisfy

$$M = \Gamma M^{T} \Gamma.$$

We call them "*odd matrices*."

There are six even matrices,

$$M_{\{30\}} = \sigma_3 \otimes \sigma_0, \quad M_{\{12\}} = \sigma_1 \otimes \sigma_2, \quad M_{\{22\}} = \sigma_2 \otimes \sigma_2,$$

For above six matrices, $M_{\{30\}}$, $M_{\{12\}}$, and $M_{\{22\}}$ anticommute with each other,

$$\{M_{\{30\}}, M_{\{12\}}\} = 0,$$

$$\{M_{\{30\}}, M_{\{22\}}\} = 0,$$

$$\{M_{\{12\}}, M_{\{22\}}\} = 0. \tag{10}$$

$M_{\{33\}}$, $M_{\{31\}}$, and $M_{\{02\}}$ anticommute with each other

$$\{M_{\{33\}}, M_{\{31\}}\} = 0,$$

$$\{M_{\{33\}}, M_{\{02\}}\} = 0,$$

$$\{M_{\{31\}}, M_{\{02\}}\} = 0. \tag{11}$$

While each of $M_{\{30\}}$, $M_{\{12\}}$, and $M_{\{22\}}$ commutate with each of $M_{\{33\}}$, $M_{\{31\}}$, and $M_{\{02\}}$. For the coefficients of the even matrices $c_{\{\alpha\beta\}}(\mathbf{k})$ in the mean-field fermion Hamiltonian, $\{\alpha\beta\}=\{30\},\{12\},\{22\},\{33\},\{31\},\{02\}$, we have

$$c_{\{\alpha\beta\}}(\mathbf{k}) = c_{\{\alpha\beta\}}(-\mathbf{k}).$$

In addition, there exist ten odd matrices:

$$M_{\{00\}} = \sigma_0 \otimes \sigma_0, \quad M_{\{10\}} = \sigma_1 \otimes \sigma_0, \quad M_{\{20\}} = \sigma_2 \otimes \sigma_0,$$

$$M_{\{03\}} = \sigma_0 \otimes \sigma_3, \quad M_{\{11\}} = \sigma_1 \otimes \sigma_1, \quad M_{\{13\}} = \sigma_1 \otimes \sigma_3,$$

$$M_{\{23\}} = \sigma_2 \otimes \sigma_3, \quad M_{\{32\}} = \sigma_3 \otimes \sigma_2, \quad M_{\{21\}} = \sigma_2 \otimes \sigma_1,$$

$$M_{\{01\}} = \sigma_0 \otimes \sigma_1.$$

For the coefficients of odd matrices $c_{\{\alpha\beta\}}(\mathbf{k})$ in the mean-field fermion Hamiltonian, we have

$$c_{\{\alpha\beta\}}(\mathbf{k}) = -c_{\{\alpha\beta\}}(-\mathbf{k}).$$

Thus for odd matrices, $c_{\{\alpha\beta\}}(\mathbf{k})$ are odd functions of k_x, k_y and are fixed to be zero at momenta (0,0), $(0,\pi)$, $(\pi,0)$, (π,π)

$$c_{\{\alpha\beta\}}(\mathbf{k}=0) = 0.$$

III. CLASSIFICATION OF *Z2A* TOPOLOGICAL ORDERS

For a generic choice of $u_{\mathbf{ij}}$ and $\eta_{\mathbf{ij}}$, the corresponding mean-field Hamiltonian 1 is gapped. Note that the energy levels of the mean-field Hamiltonian 1 appear in $(E,-E)$ pairs. The mean-field state is obtained by filling all the negative energy levels. The mean-field Hamiltonian is gapped if the minimal positive energy level is finite. The gapped mean-field Hamiltonian corresponds to a gapped *Z2A* spin liquid.

As we change the mean-field parameters $u_{\mathbf{ij}}$ and $\eta_{\mathbf{ij}}$, the mean-field energy gap (and the corresponding energy gap for the *Z2A* spin liquid) may close which indicates a quantum phase transition. Thus if two gapped regions are always separated by a gapless region, then the two gapped regions will correspond to two different quantum phases. We may say that the two quantum phases carry different topological orders.

In the following, we introduce topological indices that can be calculated for each gapped mean-field ansatz $(u_{\mathbf{ij}}, \eta_{\mathbf{ij}})$. We will show that two gapped mean-field ansatz with different topological indices cannot be smoothly deformed into each other without closing the energy gap. Therefore, the topological indices characterize different *Z2A* topological orders with translation-symmetry.

A. Topological indices

In the Sec. II, we obtained the mean-field fermion Hamiltonian in momentum space 1, which has a form $H_{\text{mean}} = H(\mathbf{k}>0)_{\text{mean}} + H(\mathbf{k}=\mathbf{0})_{\text{mean}}$. Let us diagonalize the mean-field fermion Hamiltonian at the points $\mathbf{k}>0$ as an example. Presenting

$$W(\mathbf{k})\Psi_{\mathbf{k}} = \begin{pmatrix} \alpha_{\mathbf{k}} \\ \alpha^\dagger_{-\mathbf{k}} \\ \beta_{\mathbf{k}} \\ \beta^\dagger_{-\mathbf{k}} \end{pmatrix},$$

where

$$W(\mathbf{k})U(\mathbf{k})W^\dagger(\mathbf{k}) = \begin{pmatrix} \varepsilon_1(\mathbf{k}) & 0 & 0 & 0 \\ 0 & -\varepsilon_1(\mathbf{k}) & 0 & 0 \\ 0 & 0 & \varepsilon_2(\mathbf{k}) & 0 \\ 0 & 0 & 0 & -\varepsilon_2(\mathbf{k}) \end{pmatrix}, \tag{12}$$

$\varepsilon_1(\mathbf{k})>0$, and $\varepsilon_2(\mathbf{k})>0$, we find

$$\begin{aligned} H(\mathbf{k}>0)_{\text{mean}} &= \sum_{\mathbf{k}>0} \{\varepsilon_1(\mathbf{k})\alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k}} + \varepsilon_2(\mathbf{k})\beta^\dagger_{\mathbf{k}}\beta_{\mathbf{k}} \\ &\quad - \varepsilon_1(\mathbf{k})\alpha_{-\mathbf{k}}\alpha^\dagger_{-\mathbf{k}} - \varepsilon_2(\mathbf{k})\beta_{-\mathbf{k}}\beta^\dagger_{-\mathbf{k}}\}. \end{aligned}$$

We note that both $\alpha_{\pm\mathbf{k}}$ and $\beta_{\pm\mathbf{k}}$ will annihilate the mean-field ground state $|\Psi_{\text{mean}}\rangle$,

$$\alpha_{\pm\mathbf{k}}|\Psi_{\text{mean}}\rangle = 0, \quad \beta_{\pm\mathbf{k}}|\Psi_{\text{mean}}\rangle = 0.$$

At the four $\mathbf{k}=0$ points, only the even $M_{\alpha,\beta}$ appear and we diagonalized the Hamiltonian differently. The four eigenvalues of $U(\mathbf{k})$ are given by

$$\begin{aligned} \varepsilon_{\pm\pm}(\mathbf{k}) = &\pm \sqrt{c^2_{\{30\}}(\mathbf{k}) + c^2_{\{12\}}(\mathbf{k}) + c^2_{\{22\}}(\mathbf{k})} \\ &\pm \sqrt{c^2_{\{33\}}(\mathbf{k}) + c^2_{\{31\}}(\mathbf{k}) + c^2_{\{02\}}(\mathbf{k})}, \end{aligned}$$

and

$$H(\mathbf{k}=0)_{\text{mean}} = \frac{1}{2}\sum_{\mathbf{k}=0}\{\varepsilon_{++}(\mathbf{k})\alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} + \varepsilon_{+-}(\mathbf{k})\beta_{\mathbf{k}}^{\dagger}\beta_{\mathbf{k}} + \varepsilon_{--}(\mathbf{k})\alpha_{\mathbf{k}}\alpha_{\mathbf{k}}^{\dagger} + \varepsilon_{-+}(\mathbf{k})\beta_{\mathbf{k}}\beta_{\mathbf{k}}^{\dagger}\} = \sum_{\mathbf{k}=0}\{\varepsilon_{++}(\mathbf{k})\alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} + \varepsilon_{+-}(\mathbf{k})\beta_{\mathbf{k}}^{\dagger}\beta_{\mathbf{k}}\} + \text{Const.},$$

where

$$W(\mathbf{k})U(\mathbf{k})W^{\dagger}(\mathbf{k}) = \begin{pmatrix} \varepsilon_{++}(\mathbf{k}) & 0 & 0 & 0 \\ 0 & -\varepsilon_{++}(\mathbf{k}) & 0 & 0 \\ 0 & 0 & \varepsilon_{+-}(\mathbf{k}) & 0 \\ 0 & 0 & 0 & -\varepsilon_{+-}(\mathbf{k}) \end{pmatrix}. \tag{13}$$

We note that $W(\mathbf{k})$ diagonalizes the linear combination of $M_{\{30\}}$, $M_{\{12\}}$, and $M_{\{22\}}$ in H_{mean}:

$$W(c_{\{30\}}M_{\{30\}} + c_{\{12\}}M_{\{12\}} + c_{\{22\}}M_{\{22\}})W^{\dagger} = \sqrt{c_{\{30\}}^2 + c_{\{12\}}^2 + c_{\{22\}}^2}M_{\{30\}}. \tag{14}$$

$W(\mathbf{k})$ also diagonalizes the linear combination of $M_{\{33\}}$, $M_{\{31\}}$, and $M_{\{02\}}$ in H_{mean}:

$$W(c_{\{33\}}M_{\{33\}} + c_{\{31\}}M_{\{31\}} + c_{\{02\}}M_{\{02\}})W^{\dagger} = \sqrt{c_{\{33\}}^2 + c_{\{31\}}^2 + c_{\{02\}}^2}M_{\{33\}}. \tag{15}$$

So $W(\mathbf{k})$ changes $M_{33}=\sigma_3\otimes\sigma_3$ to

$$W(\mathbf{k})(\sigma_3\otimes\sigma_3)W^{\dagger}(\mathbf{k}) = a(\mathbf{k})\sigma_3\otimes\sigma_3 + b(\mathbf{k})\sigma_0\otimes\sigma_1 + c(\mathbf{k})\sigma_3\otimes\sigma_2. \tag{16}$$

where $a^2(\mathbf{k})+b^2(\mathbf{k})+c^2(\mathbf{k})=1$.

The energy spectrum at $\mathbf{k}=0$ motivates us to present $\zeta_{\mathbf{k}}$ as the four topological indices, one for each $\mathbf{k}=0$ point:

$$\zeta_{\mathbf{k}} = 1 - \Theta[\varepsilon_{+-}(\mathbf{k})], = 1 - \Theta[c_{\{30\}}^2(\mathbf{k}) + c_{\{12\}}^2(\mathbf{k}) + c_{\{22\}}^2(\mathbf{k}) - c_{\{33\}}^2(\mathbf{k}) - c_{\{31\}}^2(\mathbf{k}) - c_{\{02\}}^2(\mathbf{k})], \tag{17}$$

where $\Theta(x)=1$ if $x>0$ and $\Theta(x)=0$ if $x<0$. If two topological ordered states have different sets of topological indices $(\zeta_{\mathbf{k}=(0,0)},\zeta_{\mathbf{k}=(\pi,0)},\zeta_{\mathbf{k}=(0,\pi)},\zeta_{\mathbf{k}=(\pi,\pi)})$, then as we deform one state smoothly into the other, some $\zeta_{\mathbf{k}}$ must change. $\zeta_{\mathbf{k}}$ can only change when

$$\sqrt{c_{\{30\}}^2(\mathbf{k}) + c_{\{12\}}^2(\mathbf{k}) + c_{\{22\}}^2(\mathbf{k})} = \sqrt{c_{\{33\}}^2(\mathbf{k}) + c_{\{31\}}^2(\mathbf{k}) + c_{\{02\}}^2(\mathbf{k})}.$$

At that point, the topological ordered state becomes gapless indicating a quantum phase transition. Therefore, there are 16 different translation invariant *Z2A* spin liquids labeled by $\zeta_{\mathbf{k}=(0,0)}$, $\zeta_{\mathbf{k}=(0,\pi)}$, $\zeta_{\mathbf{k}=(\pi,0)}$,

$$\zeta_{\mathbf{k}=(\pi,\pi)} = 1111,1100,1010,1001,0101,0011,0110,0000,1000,0100,0010,0001,1110,1101,1011,0111.$$

B. Topological degeneracy

Let's calculate topological degeneracies for different topological orders through the projective construction. Now we use $|m,n\rangle=|\Psi_{\text{mean}}^{(u_{\mathbf{ij}}^{(m,n)},\eta_{\mathbf{ij}}^{(m,n)})}\rangle(|m,n\rangle=|0,0\rangle,|0,1\rangle,|1,0\rangle,|1,1\rangle)$ to denote four degenerate ground states on a torus. Here $(u_{\mathbf{ij}}^{(m,n)},\eta_{\mathbf{ij}}^{(m,n)})$ is defined as $((-)^{ms_x(\mathbf{ij})}(-)^{ns_y(\mathbf{ij})}u_{\mathbf{ij}},(-)^{ms_x(\mathbf{ij})}(-)^{ns_y(\mathbf{ij})}\eta_{\mathbf{ij}})$. $s_{x,y}(\mathbf{ij})$ have values 0 or 1, with $s_{x,y}(\mathbf{ij})=1$ if the link ij crosses the $x=L_x$, or $y=L_y$ line(s) and $s_{x,y}(\mathbf{ij})=0$ otherwise.[24,25] In fact, the four mean-field states $|m,n\rangle=|\Psi_{\text{mean}}^{(u_{\mathbf{ij}}^{(m,n)},\eta_{\mathbf{ij}}^{(m,n)})}\rangle$ are obtained by giving the fermion wave functions $\psi(x,y)$ different boundary conditions:

$$\psi(x,y) = (-1)^m\psi(x,y+L_y),$$

$$\psi(x,y) = (-1)^n\psi(x+L_x,y). \tag{18}$$

To obtain a physical state from the mean-field ansatz $|\Psi_{\text{mean}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})}\rangle$, one needs to project $|\Psi_{\text{mean}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})}\rangle$ into the subspace with even numbers of ψ fermion per site. So the mean-field state must have even numbers of ψ fermions in order for the projection to be nonzero. The total ψ fermion number has a form $\hat{N}=N_{\mathbf{k}\neq0}+N_{\mathbf{k}=0}$ where $N_{\mathbf{k}\neq\check{0}}=\Sigma_{k\check{>}0}(\psi_{\mathbf{k}}^{\dagger}\psi_{\mathbf{k}}+\psi_{-\mathbf{k}}^{\dagger}\psi_{-\mathbf{k}})$ and $N_{\mathbf{k}=0}=\Sigma_{\mathbf{k}=0}\psi_{\mathbf{k}}^{\dagger}\psi_{\mathbf{k}}$.

We note that, for $\mathbf{k}>0$,

$$\begin{aligned}(-)^{\psi_{1,\mathbf{k}}^{\dagger}\psi_{1,\mathbf{k}}+\psi_{1,-\mathbf{k}}^{\dagger}\psi_{1,-\mathbf{k}}+\psi_{2,\mathbf{k}}^{\dagger}\psi_{2,\mathbf{k}}+\psi_{2,-\mathbf{k}}^{\dagger}\psi_{2,-\mathbf{k}}} &= (-)^{\psi_{1,\mathbf{k}}^{\dagger}\psi_{1,\mathbf{k}}-\psi_{1,-\mathbf{k}}\psi_{1,-\mathbf{k}}^{\dagger}+\psi_{2,\mathbf{k}}^{\dagger}\psi_{2,\mathbf{k}}-\psi_{2,-\mathbf{k}}\psi_{2,-\mathbf{k}}^{\dagger}} \\ &= (-)^{\psi_{1,\mathbf{k}}^{\dagger}\psi_{1,\mathbf{k}}+\psi_{1,-\mathbf{k}}\psi_{1,-\mathbf{k}}^{\dagger}+\psi_{2,\mathbf{k}}^{\dagger}\psi_{2,\mathbf{k}}+\psi_{2,-\mathbf{k}}\psi_{2,-\mathbf{k}}^{\dagger}} \\ &= (-)^{\alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+\beta_{\mathbf{k}}^{\dagger}\beta_{\mathbf{k}}+\alpha_{-\mathbf{k}}\alpha_{-\mathbf{k}}^{\dagger}+\beta_{-\mathbf{k}}\beta_{-\mathbf{k}}^{\dagger}} \\ &= (-)^{\alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+\beta_{\mathbf{k}}^{\dagger}\beta_{\mathbf{k}}+\alpha_{-\mathbf{k}}^{\dagger}\alpha_{-\mathbf{k}}+\beta_{-\mathbf{k}}^{\dagger}\beta_{-\mathbf{k}}}.\end{aligned} \tag{19}$$

Hence we have

$$\begin{aligned}(-)^{\sum_{a=1,2}(\psi_{a,\mathbf{k}}^{\dagger}\psi_{a,\mathbf{k}}+\psi_{a,-\mathbf{k}}^{\dagger}\psi_{a,-\mathbf{k}})}|\Psi_{\text{mean}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})}\rangle\rangle &= (-)^{\alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+\beta_{\mathbf{k}}^{\dagger}\beta_{\mathbf{k}}+\alpha_{-\mathbf{k}}^{\dagger}\alpha_{-\mathbf{k}}+\beta_{-\mathbf{k}}^{\dagger}\beta_{-\mathbf{k}}}|\Psi_{\text{mean}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})}\rangle \\ &= |\Psi_{\text{mean}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})}\rangle,\end{aligned} \tag{20}$$

for $\mathbf{k}>0$. We see that the total number of the ψ fermions on all the $\mathbf{k}\neq0$ orbitals is always even.

So to determine if the mean-field ground state contain even or odd number of ψ fermions, we only need to count the number ψ fermions at the four special points: (0,0), $(0,\pi)$, $(\pi,0)$, and (π,π). For $\mathbf{k}=0$,

$$(-)^{\psi_{1,\mathbf{k}}^\dagger\psi_{1,\mathbf{k}}+\psi_{2,\mathbf{k}}^\dagger\psi_{2,\mathbf{k}}}=(-)^{\psi_{1,\mathbf{k}}^\dagger\psi_{1,\mathbf{k}}-\psi_{2,\mathbf{k}}^\dagger\psi_{2,\mathbf{k}}}=(-)^{(1/2)\Psi_{\mathbf{k}}^\dagger\sigma_3\otimes\sigma_3\Psi_{\mathbf{k}}}. \tag{21}$$

Using Eq. (18), we find

$$\begin{aligned}\frac{1}{2}\Psi_{\mathbf{k}}^\dagger\sigma_3\otimes\sigma_3\Psi_{\mathbf{k}} &= \frac{1}{2}a(\mathbf{k})(\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}-\alpha_{\mathbf{k}}\alpha_{\mathbf{k}}^\dagger-\beta_{\mathbf{k}}^\dagger\beta_{\mathbf{k}}+\beta_{\mathbf{k}}\beta_{\mathbf{k}}^\dagger)\\ &\quad+\frac{1}{2}b(\mathbf{k})(\alpha_{\mathbf{k}}^\dagger\beta_{\mathbf{k}}-\alpha_{\mathbf{k}}\beta_{\mathbf{k}}^\dagger+\beta_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}-\beta_{\mathbf{k}}\alpha_{\mathbf{k}}^\dagger)\\ &\quad+i\frac{1}{2}c(\mathbf{k})(-\alpha_{\mathbf{k}}^\dagger\beta_{\mathbf{k}}-\alpha_{\mathbf{k}}\beta_{\mathbf{k}}^\dagger+\beta_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}+\beta_{\mathbf{k}}\alpha_{\mathbf{k}}^\dagger)\\ &= a(\mathbf{k})(\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}-\beta_{\mathbf{k}}^\dagger\beta_{\mathbf{k}})+b(\mathbf{k})(\alpha_{\mathbf{k}}^\dagger\beta_{\mathbf{k}}+\beta_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}})\\ &\quad+ic(\mathbf{k})(-\alpha_{\mathbf{k}}^\dagger\beta_{\mathbf{k}}+\beta_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}).\end{aligned} \tag{22}$$

We see that

$$\begin{aligned}&(-)^{\psi_{1,\mathbf{k}}^\dagger\psi_{1,\mathbf{k}}+\psi_{2,\mathbf{k}}^\dagger\psi_{2,\mathbf{k}}}\\ &\quad=e^{\pi i[a(\mathbf{k})(\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}-\beta_{\mathbf{k}}^\dagger\beta_{\mathbf{k}})+b(\mathbf{k})(\alpha_{\mathbf{k}}^\dagger\beta_{\mathbf{k}}+\beta_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}})+ic(\mathbf{k})(-\alpha_{\mathbf{k}}^\dagger\beta_{\mathbf{k}}+\beta_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}})]}.\end{aligned} \tag{23}$$

If we treat $(\alpha_{\mathbf{k}},\beta_{\mathbf{k}})$ as an isospin-1/2 doublet, then the above operator generates a 2π rotation since $a^2(\mathbf{k})+b^2(\mathbf{k})+c^2(\mathbf{k})=1$. This is consistent with the following relation:

$$\begin{aligned}\alpha_{\mathbf{k}}(-)^{\psi_{1,\mathbf{k}}^\dagger\psi_{1,\mathbf{k}}+\psi_{2,\mathbf{k}}^\dagger\psi_{2,\mathbf{k}}}&=-(-)^{\psi_{1,\mathbf{k}}^\dagger\psi_{1,\mathbf{k}}+\psi_{2,\mathbf{k}}^\dagger\psi_{2,\mathbf{k}}}\alpha_{\mathbf{k}},\\ \beta_{\mathbf{k}}(-)^{\psi_{1,\mathbf{k}}^\dagger\psi_{1,\mathbf{k}}+\psi_{2,\mathbf{k}}^\dagger\psi_{2,\mathbf{k}}}&=-(-)^{\psi_{1,\mathbf{k}}^\dagger\psi_{1,\mathbf{k}}+\psi_{2,\mathbf{k}}^\dagger\psi_{2,\mathbf{k}}}\beta_{\mathbf{k}}.\end{aligned} \tag{24}$$

Since the operator $(-)^{\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}+\beta_{\mathbf{k}}^\dagger\beta_{\mathbf{k}}}$ has the same algebra as $(-)^{\psi_{1,\mathbf{k}}^\dagger\psi_{1,\mathbf{k}}+\psi_{2,\mathbf{k}}^\dagger\psi_{2,\mathbf{k}}}$ and the two operators are equal when $a(\mathbf{k})=1$, $b(\mathbf{k})=c(\mathbf{k})=0$, we have

$$(-)^{\psi_{1,\mathbf{k}}^\dagger\psi_{1,\mathbf{k}}+\psi_{2,\mathbf{k}}^\dagger\psi_{2,\mathbf{k}}}=(-)^{\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}+\beta_{\mathbf{k}}^\dagger\beta_{\mathbf{k}}}. \tag{25}$$

Thus, the total fermion number at the four points, $\mathbf{k}=(0,0)$, $(0,\pi)$, $(\pi,0)$, and (π,π) satisfies

$$(-)^{N_{\mathbf{k=0}}}=(-)^{\sum_{\mathbf{k=0}}(\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}+\beta_{\mathbf{k}}^\dagger\beta_{\mathbf{k}})}. \tag{26}$$

For a given point of $\mathbf{k}=0$, the energies for the particles α and β are $\varepsilon_{++}(\mathbf{k})$ and $\varepsilon_{+-}(\mathbf{k})$. There are two situations:

(1) For $\varepsilon_{+-}(\mathbf{k})>0$, the **k**-orbital will be filled by 0 particle: zero α particle and zero β particle.

(2) For $\varepsilon_{+-}(\mathbf{k})<0$, the **k**-orbital will be filled by 1 particle: zero α particle and one β particle.

We find that the total number of ψ fermion at the $\mathbf{k}=0$ points and at all **k** are given by

$$N_{\mathbf{k=0}} \bmod 2 = N \bmod 2 = \sum_{k\simeq 0}\zeta_{\mathbf{k}} \bmod 2. \tag{27}$$

Note that on even-by-even (ee) lattice, all the four $\mathbf{k}=0$ points $\mathbf{k}=(0,0)$, $(0,\pi)$, $(\pi,0)$, and (π,π) are allowed. In this case N mod 2 is the sum of all four $\zeta_{\mathbf{k=0}}$ mod 2. On even-by-odd (eo) lattice, only two $\mathbf{k}=0$ points $\mathbf{k}=(0,0)$, $(\pi,0)$ are allowed. In this case $N \bmod 2=\zeta_{(0,0)}+\zeta_{(\pi,0)} \bmod 2$.

Let us use the topological order 1000 as an example to demonstrate a detailed calculation of the ground-state degeneracy. There are four degenerate mean-field ground states $|m,n\rangle=|\Psi_{\text{mean}}^{(u_{\mathbf{ij}}^{(m,n)},\eta_{\mathbf{ij}}^{(m,n)})}\rangle$, $m,n=0,1$. On an ee lattice, the state $|0,0\rangle$ has periodic boundary conditions along both x and y directions. Among the four $\mathbf{k}=0$ points, only $\mathbf{k}=(0,0)$ point has $\zeta_{\mathbf{k}}=1$ as indicated by the first 1 in the label 1000. As a result, the ground state $|0,0\rangle$ contains an odd number of fermions and is unphysical: $\mathcal{P}|0,0\rangle=0$. For the states $|0,1\rangle$, $|1,0\rangle$, and $|1,1\rangle$, the $\mathbf{k}=(0,0)$ point is not allowed. Consequently, $N=0 \bmod 2$. Hence $|0,1\rangle$, $|1,0\rangle$, and $|1,1\rangle$ are all physical states. This gave rise to three-degenerate ground states for the topological order 1000 on an (ee) lattice.

Second, we calculate the ground-state degeneracy on an eo [or odd-by-even (oe)] lattice. For the state $|0,0\rangle$, only two $\mathbf{k}=0$ points are allowed: $\mathbf{k}=(0,0)$ and $\mathbf{k}=(\pi,0)$. Due to $\zeta_{(0,0)}=1$, $|0,0\rangle$ is forbidden, $\mathcal{P}|0,0\rangle=0$. For other states $|0,1\rangle$, $|1,0\rangle$, and $|1,1\rangle$, $\mathbf{k}=(0,0)$ is not allowed on an (eo) [or (oe)] lattice. For the same reason, one obtains three-degenerate ground states $|0,1\rangle$, $|1,0\rangle$, $|1,1\rangle$ on an (eo) [or (oe)] lattice. Third, we calculate the ground-state degeneracy on an odd-by-odd (oo) lattice. For the state $|0,0\rangle$, there is only one $\mathbf{k}=0$ point: $\mathbf{k}=(0,0)$. As a result, $|0,0\rangle$ is not permitted by the projection operator, $\mathcal{P}|0,0\rangle=0$. However, without the point $\mathbf{k}=(0,0)$, other states $|0,1\rangle$, $|1,0\rangle$, and $|1,1\rangle$ are all physical. One also obtains three-degenerate ground states $|0,1\rangle$, $|1,0\rangle$, and $|1,1\rangle$ on an (oo) lattice.

By this method, we obtain topological degeneracies on different lattices for the 16 topological orders. The results are given in the following table:

	1111	1110	1101	1011	0111	1100	0011	1001	0110	1010	0101	1000	0100	0010	0001	0000
(ee)	4	3	3	3	3	4	4	4	4	4	4	3	3	3	3	4
(eo)	4	3	3	3	3	4	4	2	2	2	2	3	3	3	3	4
(oe)	4	3	3	3	3	2	2	2	2	4	4	3	3	3	3	4
(oo)	–	1	1	1	1	2	2	2	2	2	2	3	3	3	3	4

C. Crystal momenta

Next, we calculate the crystal momenta for 16 topological ordered states. Because the spin Hamiltonian is translation invariance, the ground states carry definite crystal momenta. To calculate the crystal momenta **k**, we note that the fermion wave function satisfies the (anti) periodic boundary condition.

The crystal momenta are given by

$$\hat{\mathbf{K}}|\Psi_{\text{spin}}\rangle = \sum_{\mathbf{k}} \mathbf{k}\psi_{\mathbf{k}}^{\dagger}\psi_{\mathbf{k}}|\Psi_{\text{spin}}\rangle = \sum_{\mathbf{k}\neq 0} \mathbf{k}\psi_{\mathbf{k}}^{\dagger}\psi_{\mathbf{k}}|\Psi_{\text{spin}}\rangle + \sum_{\mathbf{k}=\mathbf{0}} \mathbf{k}\psi_{\mathbf{k}}^{\dagger}\psi_{\mathbf{k}}|\Psi_{\text{spin}}\rangle. \quad (28)$$

The ground state $|\Psi^{(u_{\mathbf{ij}}^{(m,n)},\eta_{\mathbf{ij}}^{(m,n)})}_{\text{mean}}\rangle$ at $\mathbf{k}\neq 0$ has a form $\Pi_{\mathbf{k}>0}\alpha_{\mathbf{k}}\alpha_{-\mathbf{k}}\beta_{\mathbf{k}}\beta_{-\mathbf{k}}|0\rangle_{\psi}$ where $|0\rangle_{\psi}$ is the state with no ψ fermion. Thus, the total crystal momenta $\mathbf{k}$ are obtained as

$$\hat{\mathbf{K}}|\Psi_{\text{spin}}\rangle = \sum_{\mathbf{k}=\mathbf{0}} \mathbf{k}\psi_{\mathbf{k}}^{\dagger}\psi_{\mathbf{k}}|\Psi_{\text{spin}}\rangle = \sum_{\mathbf{k}=\mathbf{0}} \mathbf{k}\zeta_{\mathbf{k}}|\Psi_{\text{spin}}\rangle, \quad (29)$$

where we have used (at $\mathbf{k}=0$)

$$\psi_{\mathbf{k}}^{\dagger}\psi_{\mathbf{k}} \bmod 2 = \zeta_{\mathbf{k}}.$$

Thus, to determine the crystal momenta, we only need to focus on the cases at $\mathbf{k}=0$.

By this method, we obtain the crystal momenta on different lattices (ee, eo, oe, and oo) for 16 topological orders. The following tables show the crystal momenta of different ground states:

$\mathbf{K}(0000)$	(ee)	(eo)	(oe)	(oo)
(00)	(0,0)	(0,0)	(0,0)	(0,0)
(01)	(0,0)	(0,0)	(0,0)	(0,0)
(10)	(0,0)	(0,0)	(0,0)	(0,0)
(11)	(0,0)	(0,0)	(0,0)	(0,0)

$\mathbf{K}(0011)$	(ee)	(eo)	(oe)	(oo)
(00)	$(0,\pi)$		(0,0)	(0,0)
(01)	(0,0)		(0,0)	(0,0)
(10)	(0,0)	(0,0)	$(0,\pi)$	
(11)	(0,0)	(0,0)	(0,0)	

$\mathbf{K}(1100)$	(ee)	(eo)	(oe)	(oo)
(00)	$(0,\pi)$		$(0,\pi)$	
(01)	(0,0)		(0,0)	
(10)	(0,0)	(0,0)	(0,0)	(0,0)
(11)	(0,0)	(0,0)	(0,0)	(0,0)

$\mathbf{K}(1111)$	(ee)	(eo)	(oe)	(oo)
(00)	(0,0)	$(\pi,0)$	$(0,\pi)$	
(01)	(0,0)	$(\pi,0)$	(0,0)	
(10)	(0,0)	(0,0)	$(0,\pi)$	
(11)	(0,0)	(0,0)	(0,0)	

$\mathbf{K}(0101)$	(ee)	(eo)	(oe)	(oo)
(00)	$(\pi,0)$	(0,0)		(0,0)
(01)	(0,0)	$(\pi,0)$	(0,0)	
(10)	(0,0)	(0,0)		(0,0)
(11)	(0,0)	(0,0)	(0,0)	

$\mathbf{K}(1010)$	(ee)	(eo)	(oe)	(oo)
(00)	$(\pi,0)$	$(\pi,0)$		
(01)	(0,0)	(0,0)	(0,0)	(0,0)
(10)	(0,0)	(0,0)		
(11)	(0,0)	(0,0)	(0,0)	(0,0)

$\mathbf{K}(0110)$	(ee)	(eo)	(oe)	(oo)
(00)	(π,π)			(0,0)
(01)	(0,0)		(0,0)	
(10)	(0,0)	(0,0)		
(11)	(0,0)	(0,0)	(0,0)	(0,0)

$\mathbf{K}(1001)$	(ee)	(eo)	(oe)	(oo)
(00)	(π,π)			
(01)	(0,0)		(0,0)	(0,0)
(10)	(0,0)	(0,0)		(0,0)
(11)	(0,0)	(0,0)	(0,0)	

$\mathbf{K}(1000)$	(ee)	(eo)	(oe)	(oo)
(00)				
(01)	(0,0)	(0,0)	(0,0)	(0,0)
(10)	(0,0)	(0,0)	(0,0)	(0,0)
(11)	(0,0)	(0,0)	(0,0)	(0,0)

$\mathbf{K}(0100)$	(ee)	(eo)	(oe)	(oo)
(00)		(0,0)		(0,0)
(01)	(0,0)		(0,0)	
(10)	(0,0)	(0,0)	(0,0)	(0,0)
(11)	(0,0)	(0,0)	(0,0)	(0,0)

$\mathbf{K}(0010)$	(ee)	(eo)	(oe)	(oo)
(00)			(0,0)	(0,0)
(01)	(0,0)	(0,0)	(0,0)	(0,0)
(10)	(0,0)	(0,0)		
(11)	(0,0)	(0,0)	(0,0)	(0,0)

$\mathbf{K}(0001)$	(ee)	(eo)	(oe)	(oo)
(00)		(0,0)	(0,0)	(0,0)
(01)	(0,0)		(0,0)	(0,0)
(10)	(0,0)	(0,0)		(0,0)
(11)	(0,0)	(0,0)	(0,0)	

$\mathbf{K}(0111)$	(ee)	(eo)	(oe)	(oo)
(00)				(0,0)
(01)	(0,0)	$(\pi,0)$	(0,0)	
(10)	(0,0)	(0,0)	$(0,\pi)$	
(11)	(0,0)	(0,0)	(0,0)	

K(1011)	(ee)	(eo)	(oe)	(oo)
(00)		$(\pi,0)$		
(01)	(0,0)		(0,0)	(0,0)
(10)	(0,0)	(0,0)	$(0,\pi)$	
(11)	(0,0)	(0,0)	(0,0)	

K(1101)	(ee)	(eo)	(oe)	(oo)
(00)			$(0,\pi)$	
(01)	(0,0)	$(\pi,0)$	(0,0)	
(10)	(0,0)	(0,0)		(0,0)
(11)	(0,0)	(0,0)	(0,0)	

K(1110)	(ee)	(eo)	(oe)	(oo)
(00)		$(\pi,0)$	$(0,\pi)$	
(01)	(0,0)		(0,0)	
(10)	(0,0)	(0,0)		
(11)	(0,0)	(0,0)	(0,0)	(0,0)

IV. HOW MANY DISTINCT Z_2 TOPOLOGICAL ORDERS?

We would like to point out that the four topological indices $\zeta_{\mathbf{k}}$ at $\mathbf{k}=0$ really describe 16 classes of mean-field ansatz. It is not clear if different mean-field ansatz give rise to different many-body spin wave functions. So it is possible that the 16 sets of topological indices $\zeta_{\mathbf{k}}$ describe less than 16 classes of Z_2 topological orders.

On the other hand, if two Z_2 topological phases can be separated through measurable physical quantities, such as ground-state degeneracy and crystal momenta, then the two topological phases will be really distinct.

Using the ground-state degeneracies on different types of lattice we can group the 16 sets of topological indices $\zeta_{\mathbf{k}}$ into 7 groups: {0000}, {0001, 0010, 0100, 1000}, {0101, 1010}, {1100, 0011}, {1001, 0110}, {0111, 1011, 1101, 1110}, {1111}. Each group has the same ground-state degeneracies. So we have at least seven distinct Z_2 topological phases. If we assume that the boundary condition labeling (m,n) are not physically observable, the crystal momenta distributions cannot further separate the above seven groups into smaller groups. However, it is likely that the boundary condition labels (m,n) are physically observable by moving the unique type of fermionic excitations (the spinons) around the torus. In this case all 16 sets of topological indices $\zeta_{\mathbf{k}}$ label distinct Z_2 topological phases.

To study translation-symmetry protected topological order in three dimensions, we can still use the projective construction to construct translation symmetric Z_2 topological orders in spin-1/2 systems on cubic lattice, with mean-field fermion Hamiltonian in Eq. (1). The many-spin topological ordered state is given by $|\Psi_{\text{spin}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})}\rangle=\mathcal{P}|\Psi_{\text{mean}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})}\rangle$ where $|\Psi_{\text{mean}}^{(u_{\mathbf{ij}},\eta_{\mathbf{ij}})}\rangle$ is the ground state of H_{mean} and $\mathcal{P}=\prod_{\mathbf{i}}\frac{1+(-1)^{\psi_{\mathbf{i}}^\dagger\psi_{\mathbf{i}}}}{2}$ is the projection operator. Here $\mathbf{i}=(i_x,i_y,i_z)$ denotes a site in a cubic lattice.

Furthermore, using the same method applied to two-dimensional cases, we may define eight Z_2 topological indices $\zeta_{\mathbf{k}}=0,1$ for the Z_2 topological orders in three dimensions at $\mathbf{k}=(0,0,0)$, $(0,0,\pi)$, $(0,\pi,0)$, $(\pi,0,0)$, $(0,\pi,\pi)$, $(\pi,0,\pi)$, $(\pi,\pi,0)$, (π,π,π). The values of $\zeta_{\mathbf{k}}$ are determined from the signs of the energy spectrum at the eight $\mathbf{k}=0$ points in momentum space:

$$\zeta_{\mathbf{k}}=1-\Theta[\varepsilon_{+-}(\mathbf{k})],$$

$$=1-\Theta[c_{\{30\}}^2(\mathbf{k})+c_{\{12\}}^2(\mathbf{k})+c_{\{22\}}^2(\mathbf{k})-c_{\{33\}}^2(\mathbf{k})-c_{\{31\}}^2(\mathbf{k})-c_{\{02\}}^2(\mathbf{k})]. \tag{30}$$

where $\Theta(x)=1$ if $x>0$ and $\Theta(x)=0$ if $x<0$. Therefore, there are totally $2^8=256$ different translation invariant *Z2A* spin liquids labeled by $\zeta_{\mathbf{k}=(0,0,0)}$, $\zeta_{\mathbf{k}=(0,0,\pi)}$, $\zeta_{\mathbf{k}=(0,\pi,0)}$, $\zeta_{\mathbf{k}=(\pi,0,0)}$, $\zeta_{\mathbf{k}=(0,\pi,\pi)}$, $\zeta_{\mathbf{k}=(\pi,0,\pi)}$, $\zeta_{\mathbf{k}=(\pi,\pi,0)}$, and $\zeta_{\mathbf{k}=(\pi,\pi,\pi)}$. Similar to the cases in two dimensions, one cannot deform two states with different $\{\zeta_{\mathbf{k}}\}$ into each other without a quantum phase transition that closes the energy gap.

V. TWO EXAMPLES OF *Z2A* TOPOLOGICAL ORDERS

Let us discuss two different translation symmetric *Z2A* topological orders studied in Ref. 25 in more detail. The first one is described by the following mean-field fermion Hamiltonian[24] within the projective construction:[8]

$$H_{\text{mean}}=\sum_{\mathbf{ij}}\psi_{\mathbf{i}}^\dagger u_{\mathbf{ij}}\psi_{\mathbf{j}}+\sum_{\mathbf{i}}\psi_{\mathbf{i}}^\dagger a_{\mathbf{i}}\psi_{\mathbf{i}},$$

$$u_{\mathbf{i},\mathbf{i}+\mathbf{x}}=u_{\mathbf{i},\mathbf{i}+\mathbf{y}}=-\chi\sigma^3,$$

$$u_{\mathbf{i},\mathbf{i}+\mathbf{x}+\mathbf{y}}=\eta\sigma^1+\lambda\sigma^2,$$

$$u_{\mathbf{i},\mathbf{i}-\mathbf{x}+\mathbf{y}}=\eta\sigma^1-\lambda\sigma^2,$$

$$a_{\mathbf{i}}^1=\upsilon, \tag{31}$$

where $\psi^T=(\psi_1,\psi_2)$. The other *Z2A* spin liquid comes from an exact soluble spin-1/2 model on square lattice—the Wen-plaquette model.[16] Its Hamiltonian is $H=16g\sum_i S_i^y S_{i+\mathbf{x}}^x S_{i+\mathbf{x}+\mathbf{y}}^y S_{i+\mathbf{y}}^x$. Using projective construction, one can present a mean-field fermion Hamiltonian[24] to describe such a spin liquid:

$$H_{\text{mean}}=\sum_{\langle\mathbf{ij}\rangle}(\psi_{I,\mathbf{i}}^\dagger u_{\mathbf{ij}}^{IJ}\psi_{J,\mathbf{j}}+\psi_{I,\mathbf{i}}^\dagger\eta_{\mathbf{ij}}^{IJ}\psi_{J,\mathbf{j}}^\dagger+\text{H.c.}), \tag{32}$$

where $I,J=1,2$. It is known that the ground states for $g<0$ and $g>0$ have different symmetry protected topological orders. The ground state (*Z2A* topological order) for $g<0$ is described by mean-field ansatz $-\eta_{\mathbf{i},\mathbf{i}+\mathbf{x}}=u_{\mathbf{i},\mathbf{i}+\mathbf{x}}=1+\sigma^3$ and $-\eta_{\mathbf{i},\mathbf{i}+\mathbf{y}}=u_{\mathbf{i},\mathbf{i}+\mathbf{y}}=1-\sigma^3$. The ground state (*Z2B* topological order) for $g>0$ is described by mean-field ansatz $-\eta_{\mathbf{i},\mathbf{i}+\mathbf{x}}=u_{\mathbf{i},\mathbf{i}+\mathbf{x}}=(-)^{i_y}(1+\sigma^3)$ and $-\eta_{\mathbf{i},\mathbf{i}+\mathbf{y}}=u_{\mathbf{i},\mathbf{i}+\mathbf{y}}=1-\sigma^3$.

We like to ask: whether the topological order for *Z2A* gapped state in Eq. (31) and the topological order for the Wen-plaquette model in Eq. (32) are the same one. The four Z_2 topological variables $\zeta_{\mathbf{k}=0}$ can help us to answer the question.

For the *Z2A* gapped state, a mean-field fermion Hamiltonian in momentum space becomes

$$H_{\text{mean}}(\mathbf{k}) = \sum_{\mathbf{k}} \Psi_{\mathbf{k}}^{\dagger} U(\mathbf{k}) \Psi_{\mathbf{k}} + \text{H.c.}, \tag{33}$$

where

$$U(\mathbf{k}) = \sum_{\{\alpha,\beta\}} c_{\{\alpha\beta\}}(\mathbf{k}) M_{\{\alpha\beta\}},$$

$$c_{\{33\}} = \cos k_x + \cos k_y,$$

$$c_{\{31\}} = \eta[\cos(k_x + k_y) + \cos(k_x - k_y)] + \upsilon,$$

$$c_{\{02\}} = \lambda[\cos(k_x + k_y) - \cos(k_x - k_y)].$$

Because $M_{\{33\}}$, $M_{\{31\}}$, and $M_{\{02\}}$ make up of an anticommuting basis, we have

$$\begin{aligned}\varepsilon_{+-}(\mathbf{k}) &= 0 - \sqrt{c_{\{33\}}^2(\mathbf{k}) + c_{\{31\}}^2(\mathbf{k}) + c_{\{02\}}^2(\mathbf{k})} \\ &= -\sqrt{(\cos k_x + \cos k_y)^2 + \{\eta[\cos(k_x + k_y) + \cos(k_x - k_y)] + \upsilon\}^2 + \{\lambda[\cos(k_x + k_y) - \cos(k_x - k_y)]\}^2}.\end{aligned}$$

For the *Z2A* gapped state, one has

$$\zeta_{(0,0)} = 1, \quad \zeta_{(0,\pi)} = 1, \quad \zeta_{(\pi,0)} = 1, \quad \zeta_{(\pi,\pi)} = 1.$$

It belongs to the 1111 type of the topological order. The topological degeneracy is 4, 4, and 4, for even-by-even, even-by-odd, and odd-by-even, respectively. On odd-by-odd lattices, the state has no energy gap.

Another example of topological order is the Wen-plaquette model. The mean-field ansatz in momentum space now becomes

$$U(\mathbf{k}) = \sum_{\{\alpha,\beta\}} c_{\{\alpha\beta\}}(\mathbf{k}) M_{\{\alpha\beta\}},$$

$$c_{\{30\}} = \cos k_x + \cos k_y,$$

$$c_{\{33\}} = \cos k_x - \cos k_y,$$

$$c_{\{20\}} = \sin k_x + \sin k_y,$$

$$c_{\{23\}} = \sin k_x - \sin k_y.$$

There are two even matrices in the ansatz, $M_{\{30\}}$ and $M_{\{33\}}$. We have

$$\varepsilon_{+-}(\mathbf{k}=0) = |\cos k_x + \cos k_y| - |\cos k_x - \cos k_y|.$$

For the topological order of the Wen-plaquette model, one has

$$\zeta_{(0,0)} = 0, \quad \zeta_{(0,\pi)} = 1, \quad \zeta_{(\pi,0)} = 1, \quad \zeta_{(\pi,\pi)} = 0.$$

It belongs to the type of the topological order. The topological degeneracy is 4, 2, 2, and 2 for (ee), (eo), (oe), and (oo) lattices. Because topological order in the toric-code model with translation invariance[12] and that in the Wen-plaquette model are equivalent, one can use the same wave function to describe the toric-code model.

Then when one changes the *Z2A* gapped state denoted by 1111, into the topological order of the Wen-plaquette model denoted by, quantum phase transition occurs with emergent massless fermion at $\mathbf{k}=(0,0)$ and $\mathbf{k}=(\pi,\pi)$.

VI. NON-ABELIAN TOPOLOGICAL STATES

We like to point out that for the four *Z2A* topological states described by $\{\zeta_{\mathbf{k}}\}$=1000,0100,0010,0001, the corresponding mean-field Hamiltonian describes a superconducting state whose band structure has an odd winding number.[26,27] So those four *Z2A* topological states are closely related to the topological spin liquid state obtained by the projection of p_x+ip_y SC states.[27] As a result, the four *Z2A* topological states have a topological order described by Ising topological quantum field theory (which is the same topological quantum field theory describing the non-Abelian Paffien FQH state at $\nu=1/2$).[28–32] This is consistent with the fact that those four *Z2A* topological states all have three-degenerate ground states on torus and do not have time reversal symmetry. Therefore the four *Z2A* topological states are non-Abelian states with excitations that carry non-Abelian statistics described by Ising topological quantum field theory.

We believe that the four *Z2A* topological states described by $\{\zeta_{\mathbf{k}}\}$=0111,1011,1101,1110 are also non-Abelian states described by Ising topological quantum field theory. However, 0111, 1011,1101, and 1110 states are different from the 1000, 0100, 0010, and 0001 states since they have different ground-state degeneracy on (oo) lattices. Some concrete constructions of those non-Abelian topological states and other topological states discussed in this paper are given in Appendix B.

We also like to point out that the mean-field Hamiltonian is a Hamiltonian for a superconductor. The Z_2 topological indices $\zeta_{\mathbf{k}}$ provide a classification of translation invariant two-dimensional (2D) topological superconductors as discussed in Ref. 33.

VII. CONCLUSION

In this paper, we study topological phases that have the translation-symmetry using the projective approach. We concentrated on a class of topological phases described by $Z2A$ PSG. We find that 2D $Z2A$ topological phases on square lattice can be further divided into 16 classes, which can be described by four Z_2 topological variables $\zeta_{\mathbf{k}}=0,1$ at the four points $\mathbf{k}=(0,0)$, $(0,\pi)$, $(\pi,0)$, and (π,π). Through the projected SC wave functions, we obtain the topological degeneracies and the crystal momenta for those 16 classes of topological states. This allows us to identify the topological phase of the Wen's plaquette model as the 0110 $Z2A$ topological order. In addition, it is predicted that massless fermionic excitations appear at the quantum phase transition between different topological orders with translation invariance.

What are the effective topological field theories for those different symmetry protected Z_2 topological orders on lattices? In Ref. 25, it is pointed out that the continuum effective theories for all the two-dimensional Z_2 topological orders are same $U(1)\times U(1)$ mutual Chern-Simons (MCS) theory. However, the lattice symmetry (such as the translation-symmetry) have different realizations in the $U(1)\times U(1)$ MCS theory. In particular, the 1111 type and 0110 type Z2 topological orders and the corresponding realization of the lattice translation-symmetry in the MCS theory are discussed in detail in Ref. 25. We believe that the symmetry protect Z_2 topological ordered states discussed here are also described $U(1)\times U(1)$ MCS theory with different realization of lattice translation-symmetry. The results obtained in this paper apply to any lattices (since they all have translation-symmetry). Square lattice, triangle lattice, etc., have additional and different rotation and reflection symmetries. We can have richer symmetry protected topological orders protected by those additional symmetries. Many of those symmetry protected topological orders can be described by the projective symmetry group.[24] There are also symmetry protected topological orders that cannot be described by projective symmetry group as we have seen in this paper. Those additional topological orders may be described by different realizations of lattice rotation/reflection symmetries in the $U(1)\times U(1)$ MCS theory.

ACKNOWLEDGMENTS

This research is supported by NSF Grant No. DMR-0706078, NFSC Grant No. 10228408, and NFSC Grant No. 10874017.

APPENDIX A: DEFINITION OF ANTICOMMUTING BASIS

In this appendix we define anticommuting basis. An anticommuting basis $(M_{\{\alpha\beta\}},M_{\{\alpha'\beta'\}},M_{\{\alpha''\beta''\}},\ldots)$ is a maximum set for several 4×4 matrices which anticommute each other

$$\{M_{\{\alpha\beta\}},M_{\{\alpha'\beta'\}}\}=0,$$

$$\{M_{\{\alpha'\beta'\}},M_{\{\alpha''\beta''\}}\}=0,$$

$$\{M_{\{\alpha\beta\}},M_{\{\alpha''\beta''\}}\}=0,$$

$$\cdots.$$

If a matrix can be decomposed into an anticommuting basis,

$$U=\sum_{\{\alpha,\beta\}} c_{\{\alpha\beta\}}M_{\{\alpha\beta\}},$$

one has the determinant of the matrix U as

$$\det U=[c^2_{\{\alpha\beta\}}+c^2_{\{\alpha'\beta'\}}+c^2_{\{\alpha''\beta''\}}+\cdots]^2.$$

When one add another matrix $U'=\Sigma_{\{\alpha,\beta\}}\tilde{c}_{\{\alpha\beta\}}M_{\{\alpha\beta\}}$ to U, we have the determinant as

$$\det(U+U')=[(c_{\{\alpha\beta\}}+\tilde{c}_{\{\alpha\beta\}})^2+(c_{\{\alpha'\beta'\}}+\tilde{c}_{\{\alpha'\beta'\}})^2 + (c_{\{\alpha''\beta''\}}+\tilde{c}_{\{\alpha''\beta''\}})^2+\cdots]^2. \tag{A1}$$

A famous example is Dirac/Clifford algebra, of which five γ matrices, $M_{\{33\}}$, $M_{\{13\}}$, $M_{\{11\}}$, $M_{\{02\}}$, and $M_{\{01\}}$ make up an anticommuting basis. For a matrix of the form

$$U=\sum_{\{\alpha,\beta\}} c_{\{\alpha\beta\}}M_{\{\alpha\beta\}},\quad \{\alpha\beta\}=\{33\},\{13\},\{11\},\{02\},\{01\},$$

the eigenvalue of U are

$$\pm\sqrt{c^2_{\{33\}}+c^2_{\{13\}}+c^2_{\{11\}}+c^2_{\{02\}}+c^2_{\{01\}}},$$

and the determinant of U is

$$\det U=(c^2_{\{33\}}+c^2_{\{13\}}+c^2_{\{11\}}+c^2_{\{02\}}+c^2_{\{01\}})^2.$$

$M_{\{30\}}$, $M_{\{12\}}$, and $M_{\{22\}}$ make up of an anticommuting basis. For a matrix of the form

$$U=\sum_{\{\alpha,\beta\}} c_{\{\alpha\beta\}}M_{\{\alpha\beta\}},\quad \{\alpha\beta\}=\{30\},\{12\},\{22\},$$

the eigenvalues of U are

$$\pm\sqrt{c^2_{\{30\}}+c^2_{\{12\}}+c^2_{\{22\}}}.$$

U can be diagonalized $U=c_{\{30\}}M_{\{30\}}+c_{\{12\}}M_{\{12\}}+c_{\{22\}}M_{\{22\}}$ into a 4×4 matrix as

$$\sqrt{c^2_{\{30\}}+c^2_{\{12\}}+c^2_{\{22\}}}\times\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}.$$

$M_{\{33\}}$, $M_{\{31\}}$, and $M_{\{02\}}$ make up of another anticommuting basis. One can diagonalize $U=c_{\{33\}}M_{\{33\}}+c_{\{31\}}M_{\{31\}}+c_{\{02\}}M_{\{02\}}$, into another 4×4 matrix as

$$\sqrt{c^2_{\{33\}}+c^2_{\{31\}}+c^2_{\{02\}}}\times\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}.$$

APPENDIX B: THE EXAMPLES OF THE ANSATZ OF DIFFERENT *Z2A* TOPOLOGICAL ORDERS

In this part we give one example for each type of topological orders. The ansatz $U(\mathbf{k})$ in momentum space of the 16 classes topological orders are given by

$$0000:\begin{pmatrix} C_1 & 0 \\ 0 & C_1 \end{pmatrix},\quad 0011:\begin{pmatrix} \sigma^3 & 0 \\ 0 & B_x \end{pmatrix},$$

$$0101:\begin{pmatrix} \sigma^3 & 0 \\ 0 & B_y \end{pmatrix},\quad 0110:\begin{pmatrix} \sigma^3 & 0 \\ 0 & B_{x+y} \end{pmatrix},$$

$$1111:\begin{pmatrix} C_1 & 0 \\ 0 & -C_1 \end{pmatrix},\quad 1100:\begin{pmatrix} \sigma^3 & 0 \\ 0 & -B_x \end{pmatrix},$$

$$1010:\begin{pmatrix} \sigma^3 & 0 \\ 0 & -B_y \end{pmatrix},\quad 1001:\begin{pmatrix} \sigma^3 & 0 \\ 0 & -B_{x+y} \end{pmatrix},$$

$$1110:\begin{pmatrix} \sigma^3 & 0 \\ 0 & -C_1 \end{pmatrix},\quad 1011:\begin{pmatrix} \sigma^3 & 0 \\ 0 & -C_2 \end{pmatrix},$$

$$0111:\begin{pmatrix} \sigma^3 & 0 \\ 0 & -C_3 \end{pmatrix},\quad 1101:\begin{pmatrix} \sigma^3 & 0 \\ 0 & -C_4 \end{pmatrix},$$

$$0001:\begin{pmatrix} \sigma^3 & 0 \\ 0 & C_1 \end{pmatrix},\quad 0100:\begin{pmatrix} \sigma^3 & 0 \\ 0 & C_2 \end{pmatrix},$$

$$1000:\begin{pmatrix} \sigma^3 & 0 \\ 0 & C_3 \end{pmatrix},\quad 0010:\begin{pmatrix} \sigma^3 & 0 \\ 0 & C_4 \end{pmatrix}.$$

The parameters above are defined as

$$\sigma^3 = \begin{pmatrix} 1+\frac{1}{4}\cos k_x + \frac{1}{4}\cos k_y & 0 \\ 0 & -1-\frac{1}{4}\cos k_x - \frac{1}{4}\cos k_y \end{pmatrix},$$

$$B_x = \begin{pmatrix} \cos k_x & i\sin k_x \\ -i\sin k_x & -\cos k_x \end{pmatrix},$$

$$B_y = \begin{pmatrix} \cos k_y & i\sin k_y \\ -i\sin k_y & -\cos k_y \end{pmatrix},$$

$$B_{x+y} = \begin{pmatrix} \cos(k_x+k_y) & i\sin(k_x+k_y) \\ -i\sin(k_x+k_y) & -\cos(k_x+k_y) \end{pmatrix},$$

$$C_1 = \begin{pmatrix} \cos k_x - \frac{1}{2}\cos(k_x+k_y) + \cos k_y & \sin k_x + i\sin k_y \\ \sin k_x - i\sin k_y & -\cos k_x + \frac{1}{2}\cos(k_x+k_y) - \cos k_y \end{pmatrix},$$

$$C_2 = \begin{pmatrix} \cos k_x - \frac{1}{2}\cos(k_x+k_y) - \cos k_y & \sin k_x + i\sin k_y \\ \sin k_x - i\sin k_y & -\cos k_x - \frac{1}{2}\cos(k_x+k_y) + \cos k_y \end{pmatrix},$$

$$C_3 = \begin{pmatrix} \cos k_x + \cos(k_x+k_y) + \frac{1}{2}\cos k_y & \sin k_x + i\sin k_y \\ \sin k_x - i\sin k_y & -\cos k_x - \cos(k_x+k_y) - \frac{1}{2}\cos k_y \end{pmatrix},$$

$$C_4 = \begin{pmatrix} \cos k_x + \cos(k_x+k_y) - \frac{1}{2}\cos k_y & \sin k_x + i\sin k_y \\ \sin k_x - i\sin k_y & -\cos k_x - \cos(k_x+k_y) + \frac{1}{2}\cos k_y \end{pmatrix}.$$

One can check above results by the following table:

	(0,0)	$(0,\pi)$	$(\pi,0)$	(π,π)
$1+1/4\cos k_x+1/4\cos k_y$	>0	>0	>0	>0
$\cos k_x$	>0	>0	<0	<0
$\cos k_y$	>0	<0	>0	<0
$\cos(k_x+k_y)$	>0	<0	<0	>0
$\cos k_x+\cos k_y-1/2\cos(k_x+k_y)$	>0	>0	>0	<0
$\cos k_x-\cos k_y-1/2\cos(k_x+k_y)$	<0	>0	<0	<0
$\cos k_x+\cos(k_x+k_y)+1/2\cos k_y$	>0	<0	<0	<0
$\cos k_x+\cos(k_x+k_y)-1/2\cos k_y$	>0	>0	<0	>0

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